On alternating closed braids

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Abstract

We introduce a numerical invariant called the braid alternation number that measures how far a link is from being an alternating closed braid. This invariant resembles the alternation number, which was previously introduced by the second author. However, these invariants are not equal, even for alternating links.

We study the relation of this invariant with others and calculate this invariant for some infinite knot families. In particular, we show arbitrarily large gaps between the braid alternation number and the alternation and unknotting numbers. Furthermore, we estimate the braid alternation number for prime knots with nine crossings or less.

1 Introduction

There exist numerical invariants that measure how far a link is from the set of alternating links. In particular, the second author introduced the alternation number of a link, [16]. The alternation number of a link diagram D is the minimum number of crossing changes necessary to transform D into some (possibly non-alternating) diagram of an alternating link. The *alternation number* of a link L, denoted by $alt(L)$, is the minimum alternation number of any diagram of L. The alternation number of L is also the Gordian distance from L to the set of alternating links. Another numerical invariant was introduced by Adams et al [3]. The dealternating number of a link diagram D is the minimum number of crossing changes necessary to transform D into an alternating diagram. The *dealternating number* of a link L, denoted by $dalt(L)$, is the minimum dealternating number of any diagram of L. A link with dealternating number k is also called *k-almost alternating*. It is immediate from their definitions that $alt(L) \leq dalt(L)$ for any link L.

By Alexander's theorem, it is known that any link can be presented as a closed braid [4]; however, there are alternating links that cannot be presented as alternating closed braids [6]. Then a natural question is the following, what is the minimal number of crossings changes needed to transform a link into an alternating closed braid? In order to answer this question, for a link L, we introduce the *braid alternation* and the *braid dealternating* numbers, which are link invariants that measure how far is a link from being an alternating closed braid and are denoted by $Balt(L)$ and $Bdalt(L)$, respectively. The precise definitions are given in Section 3. Furthermore, we show that these invariants are independent of the classic definitions for links. We study the relation between these invariants showing that all of them are pairwise distinct. The value of these invariants has been obtained for some knot families.

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This paper is organized as follows: In Section 2, we remark some definitions concerning braids and links, and the relation between them. After that, in Section 3, the braid alternation number and the braid dealternating number are defined. In Section 4, these invariants are calculated for some links families, and a large gap between their values is shown. In Section 5, we give a table with the braid alternation and braid dealternating numbers of prime knots up to 9 crossings.

2 Braids and links

A braid on n-strands is an element of the n-braid group B_n which can be expressed as a word in generators $\sigma_1, \cdots, \sigma_{n-1}$ where σ_i is the braid involving a single crossing of the ith and $i+1$st strands. They are related by the following relations:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| > 1$;
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, if $i = 1,\ldots,n-2$.

A braid is said to be *alternating* if its even-numbered generators have the opposite sign to its odd-numbered generators. A braid β on n strands is said to be *homogeneous* if for every $i = 1,2,\ldots,n-1$ the generator σ_i appears in β if and only if σ_i^{-1} does not appear. Note that all alternating braids in which every generator appears at least once are homogeneous.

We call closed braid diagrams to the diagrams of a link that are presented as the closure of a braid. A link is said to be *alternating* if and only if it admits an alternating link diagram, i. e. a diagram with alternating underpasses and overpasses. The trivial link is an example of an alternating link. Note that an alternating closed braid represents an alternating link, but not all alternating links can be presented as an alternating closed braid.

As a consequence of Alexander's theorem, some algorithms have surfaced for obtaining a closed braid from a given link diagram [26, 25, 14]. In particular, due to the algorithm from S. Yamada, we note the following.

Theorem 2.1. *(cf. Theorem 1.2.2 of [14]) Any link diagram D can be deformed into a closed braid diagram $\hat{B}$ such that every crossing in D remains in $\hat{B}$.*

Proof. This fact is a consequence of the proof given for Theorem 1.2.2 in [14] since any link diagram is deformed through a finite sequence of concentric deformations such that the connecting arcs which represent the crossings are preserved after each concentric deformation until it becomes a system of concentric Seifert circles.

The effect of a concentric deformation in the neighborhood of a crossing due to a concentric deformation is shown in Figure 1. In particular, (a), (b), and (c) represent the neighborhood of a crossing in the link diagram, and (d), (e), and (f) represent their corresponding Seifert circles and connecting arcs. Figures 1 (a) and 1 (d) are before performing the concentric deformation. So, any link diagram can be deformed into a closed braid preserving all the crossings of the original diagram. □

Figure 1: Diagrams (on the left) and their corresponding Seifert circles with connecting arcs (on the right) before and after a concentric deformation. The original crossing and the corresponding connecting arcs are highlighted in blue.

A classical invariant that is hard for computing is the following. The *unknotting number* of a link L, denoted by $u(L)$, is defined to be the minimum number of crossing changes required to convert L into the trivial link overall link diagrams representing L. There is no algorithm to compute it for a given link until now.

A consequence of Theorem 2.1 is that if the condition 'overall link diagrams'on the previous definition is restricted to 'overall closed braid diagrams', then the unknotting number of a link L can still be obtained. Therefore, some invariants such as the unknotting number and the alternation number can be estimated over the restricted set of closed braid diagrams.

Each link can be presented through diagrams with a distinct number of crossings. A crossing is *reducible* if there is a circle in the projection plane meeting the diagram transversely at that crossing, but not meeting the diagram at any other point, see Figure 2. A diagram is called *reduced* if it is non-split and none of its crossings are reducible. A braid is called reduced if its closure is a reduced link diagram. A reduced alternating diagram of a link L has the minimal number of crossings for its diagrams. In [17], a table of prime links enumerated by a canonical order up to ten crossings is given, in which each link is presented as a reduced closed braid.

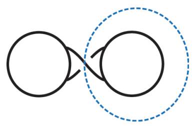
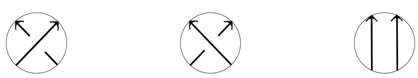


Figure 2: A reducible crossing.

The *Alexander polynomial* $\Delta(L;t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is an isotopy invariant of an oriented link L [5]. It can be computed by the following recursive relations:

1. $\Delta(L_+;t) - \Delta(L_-;t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0;t)$,
2. $\Delta(U;t) = 1$,

where U is the trivial knot; (L_+, L_-, L_0) is a skein triple of oriented links which are the same, except in a crossing neighborhood where they look as shown in Figure 3.



(a) L_+ (b) L_- (c) L_0

Figure 3: Skein triple.

Let $\bar{\sigma}(L)$ be the signature of a link L [21], where the right-hand trefoil knot has signature 2. The signature can be calculated by skein relation.

Lemma 2.2. *[8] Suppose K is a knot (but not a link), and D is a regular diagram for K. Then* $\bar{\sigma}(K)$ *can be determined by means of the following three axioms.*

- *If U is the trivial knot, then* $\bar{\sigma}(U) = 0$.
- *If* D_+ *and* D_- *are the skein diagrams, then* $\bar{\sigma}(D_-) \leq \bar{\sigma}(D_+) \leq \bar{\sigma}(D_-) + 2$.
- *If* $\Delta(K;t)$ *is the Alexander polynomial of K, then* $sign(\Delta(K;-1)) = (-1)^{\frac{\bar{\sigma}(K)}{2}}$.

Lemma 2.3. *[20] If* $u(K)$ *is the unknotting number of the knot K, then* $|\bar{\sigma}(K)| \leq 2u(K)$.

For a pair of positive integers (p,q), we define a torus knot or link of type (p,q) as the closure of the p-braid $B = (\sigma_{p-1}\sigma_{p-2}\ldots\sigma_2\sigma_1)^q$, which we denote by $T(p,q)$. The signature of a torus knot can be calculated recursively, [11]. In particular, $\bar{\sigma}(T(2,q)) = q-1$. The unknotting number of the torus knot $T(p,q)$ is determined by Kronheimer and Mrowka [18]:

$$u(T(p,q)) = \frac{(p-1)(q-1)}{2}.$$

3 Braid alternation number and braid dealternating number

In order to measure the Gordian distance between a link and the set of links that can be presented as alternating closed braids, we introduce the following new numerical invariant.

Definition 3.1. *The braid alternation number of a closed braid diagram D is the minimum number of crossing changes necessary to transform D into some (possibly non-alternating) diagram of an alternating closed braid. The braid alternation number of a link L, denoted by* $Balt(L)$*, is the minimum braid alternation number of any closed braid diagram of L.*

By definition, the alternation number is smaller than or equal to the braid alternation number. But, we affirm that the alternation number is not equal to the braid alternation number since the alternating links have alternation number zero, and some of them cannot be presented as alternating closed braids. One way to prove this fact is by Lemma 3.2. A link is called *fibered* if its exterior is a surface bundle over S^1 such that each fiber is a Seifert surface for the link.

Lemma 3.2. *Let L be a non-split link. If* $Balt(L) = 0$*, then L is fibered.*

Proof. Let L be a non-split link with $Balt(L) = 0$. Then L can be presented as the closure of an alternating braid. Since any alternating braid that does not close to a split link is homogeneous, L is fibered due to Stallings [24]. □

In particular, we can see that the braid alternation number is not equal to the alternation number as follows. Since the knot 5_2 is alternating, then $alt(5_2) = 0$. However, as the knot 5_2 is non-fibered, from Lemma 3.2, it follows that $Balt(5_2) \neq 0$. It is well known that an alternating link is fibered if and only if the leading coefficient of its Alexander polynomial is ± 1 (see [22, 23] cf. [6]). Lemma 3.2 implies that for a non-split alternating link L we have that $Balt(L) \neq 0$ if the leading coefficient of its Alexander polynomial is different to ± 1.

Another form of determining whether the prime links do not have an alternating closed braid diagram is by using reduced closed braid diagrams, as shown in Lemma 3.3.

Lemma 3.3. *Every prime alternating closed braid D is deformed into a reduced alternating closed braid.*

Proof. Assume an alternating closed braid has a reducible crossing point p, see Figure 4. Let $p = \sigma_i^{\pm 1}$. Since p is reducible, the closed braid D does not contain any other $\sigma_i^{\pm 1}$. Let D_1 and D_2 be the closed braids obtained from the closed braid D by splicing at p. Since D is prime, D_1 or D_2 is a closed braid of the trivial knot. Say D_2 is a trivial knot diagram. Then D_1 is an alternating closed braid deformed from D. By continuing this process on reducible crossings, we obtain a reduced alternating closed braid. □

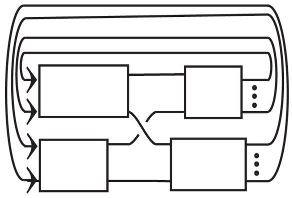


Figure 4: An alternating closed braid with a reducible crossing point.

Note 3.4. *A consequence of Lemma 3.3, due to the solution of the Tait conjecture, is the following: If K is a prime knot with $Balt(K) = 0$, there exists an alternating closed braid diagram D of K such that $c(D) = c(K)$, where $c(D)$ and $c(K)$ are the crossing numbers of D and K, respectively.*

Consequently, if a prime knot K does not have a reduced alternating closed braid diagram, $Balt(K) \neq 0$. Therefore, since the knot 5_2 does not have a reduced alternating closed braid diagram [17], it is not possible to present it as an alternating closed braid.

Note 3.5. *The converse of Lemma 3.2 does not hold. An example of this fact is given when considering the knot 10_{60}, which is a prime alternating fibered knot. However, since this knot does not have an alternating closed braid diagram with ten crossings, see [17] and [9], it follows that $Balt(10_{60}) \neq 0$.*

The braid alternation number resembles the alternation number. Similarly, we define an invariant that resembles the dealternating number.

Definition 3.6. *The braid dealternating number of a closed braid diagram D is the minimum number of crossing changes necessary to transform D into an alternating closed braid diagram. The braid dealternating number of a link L, denoted by $Bdalt(L)$, is the minimum dealternating number of any closed braid diagram of L.*

It follows that $Balt(L) \leq Bdalt(L)$. The braid dealternating number is not equal to dealternating number, as before, if we consider the alternating knot 5_2, we note that $dalt(5_2) = 0$ and $Bdalt(5_2) \geq 1$ since $Balt(5_2) \geq 1$. The following relations between these numerical invariants follow from their definitions.

$$\begin{array}{ccc} dalt(L) & \leq & Bdalt(L) \\ \vee\!\mid & & \vee\!\mid \\ alt(L) & \leq & Balt(L) \end{array} \tag{1}$$

By Lemma 3.2, there exist links whose alternation numbers are different from their braid alternation numbers, however also there exist links with the alternation number equal to the braid alternation number, as Theorem 4.1 shows.

A property of the braid alternation number is that it is subadditive under connected sum.

Proposition 3.7. *Let L_1 and L_2 be two links then $Balt(L_1\#L_2) \leq Balt(L_1) + Balt(L_2)$, where # denotes a connected sum of L_1 and L_2.*

Proof. Let B_1^a and B_2^b be two closed braid diagrams of links L_1 and L_2, respectively, such that $Balt(B_1^a) = Balt(L_1) = a$ and $Balt(B_2^b) = Balt(L_2) = b$ where $a, b \in \mathbb{N} \cup \{0\}$. Let B_1 and B_2 be alternating closed braids obtained from B_1^a and B_2^b by a crossing changes and b crossing changes, respectively. Let $L_1\#L_2$ be a connected sum of L_1 and L_2, and let $B_1^a\#B_2^b$ be a connected sum of B_1^a and B_2^b such that $B_1^a\#B_2^b$ is a diagram of $L_1\#L_2$. The diagram $B_1^a\#B_2^b$ can be presented as a closed braid B^{a+b}, which through $a+b$ crossing changes is transformed into an alternating closed braid of $B_1\#B_2$, such that the crossings outside of B_1 and B_2 are one or two crossings with signs adequately chosen depending on the sign of the generators of B_1 and B_2, see Figure 5. It follows that $Balt(B^{a+b}) = Balt(B_1^a) + Balt(B_2^b)$. Then, $Balt(L_1\#L_2) \leq Balt(L_1) + Balt(L_2)$. □

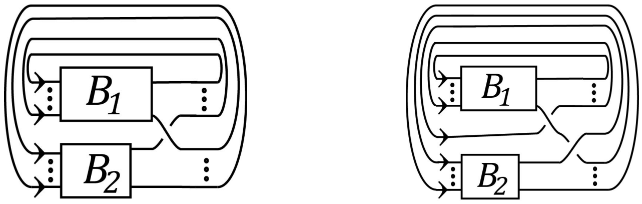


Figure 5: Alternating connected sum of two alternating closed braids. In the first case, the last generator of B_1 has a negative sign equal to the sign of the first generator of B_2. In the second case, these generators have different signs.

In the next section, we shall see that the inequality in Proposition 3.7 becomes equality for particular links. Besides, we shall estimate the invariants in Inequality (1) for some link families.

4 Link families

In the previous section, we saw that the braid alternation and braid dealternating numbers are distinct from the alternation, dealternating numbers, respectively. However, there exist links with the same value for all these invariants. In particular, for a family of closed 3-braids, we have the following theorem.

Theorem 4.1. *Let K be a knot such that is the closure of a 3-braid of the form*

$$\Delta^{2n} \prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i}$$

where $n \geq 0$, $p_i, q_i \geq 2$ for $i = 1, 2, ..., r$, and $\Delta = \sigma_1\sigma_2\sigma_1$.

Then we have $alt(K) = Balt(K) = dalt(K) = Bdalt(K) = n + r - 1$.

Proof. Abe and Kishimoto in [2, Thm 3.1] showed that $alt(K) = dalt(K) = n + r - 1$. Then, due to Inequality (1), we have that $Bdalt(K) \geq Balt(K) \geq n + r - 1$. On the other hand, following their idea, the braid $\Delta^{2n} \prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i}$ can be written as $\prod_{i=1}^{n+r} \sigma_1^{l_i} \sigma_2^{m_i}$, where

$l_i, m_i \in \mathbb{N}$, by the following equalities.

$$\Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i} = (\sigma_2\sigma_1^2\sigma_2)^n\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}$$
$$= \sigma_2(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}.$$

If $r = 1$, then

$$\Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i} = (\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1+1}.$$

If $r \geq 2$, then

$$\Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i} = (\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1}\left[\prod_{i=2}^{r-1}\sigma_1^{p_i}\sigma_2^{q_i}\right]\sigma_1^{p_r}\sigma_2^{q_r+1}.$$

A diagram of the closure of $\Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}$ with $n+r$ factors $\sigma_1^{l_i}\sigma_2^{m_i}$ is given in Figure 6 (a). We can deform this diagram to obtain a closed braid diagram D such that $Bdalt(D) = n+r-1$, as shown in Figure 6 (b). Performing alternatively $n+r-1$ crossing changes in the modified strand, we obtain a closed alternating braid. Therefore, $Bdalt(K) \leq n+r-1$ and consequently $Balt(K) = Bdalt(K) = n+r-1$. □

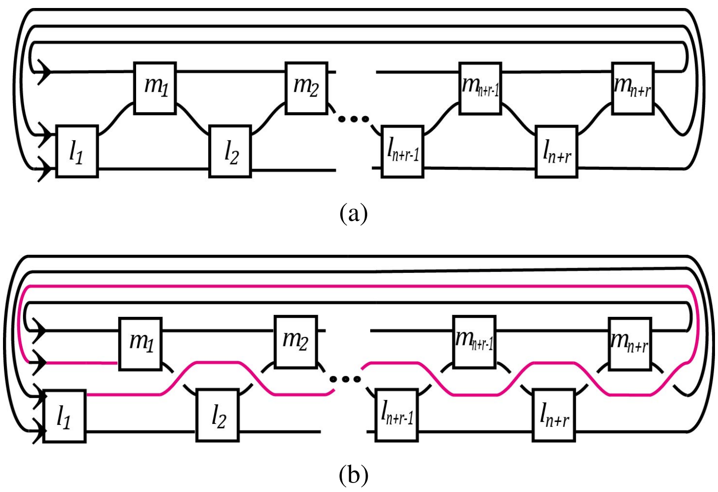


Figure 6: Two equivalents closed braids

On the other hand, the first author in [12] gave a family of hyperbolic prime knots, denoted by $\mathcal{D}$, where the difference between the alternation and the dealternating numbers of each knot is arbitrarily large. The family was constructed by concatenating a 3-braid of form $\sigma_2^{2l+1}(\sigma_2\sigma_1\sigma_2)^{2n}$ with $l+1, n \in \mathbb{N}$ and a 3-tangle, denoted by c, as shown in Figure 7. The upper bound of the alternation number was obtained performing a crossing change in 3-tangle c. The diagram after that crossing change is a diagram of either a torus knot of two strands or the trivial knot. Two equivalent diagrams of a knot diagram in $\mathcal{D}$, after that crossing change, are shown in Figure 8.

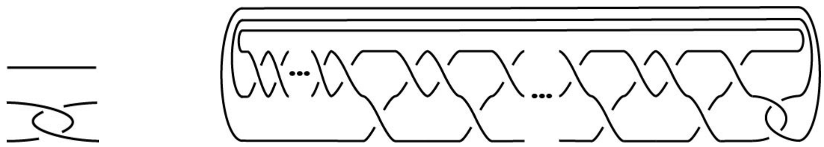


Figure 7: On the left 3-tangle c and on the right a knot diagram of $\mathcal{D}$.

Lemma 4.2. *[12] For each $n \in \mathbb{N}$, there exists an infinite knot family $\mathcal{D}_n$ in $\mathcal{D}$ such that if $K \in \mathcal{D}_n$, then $alt(K) = 1$ and $dalt(K) = n$.*

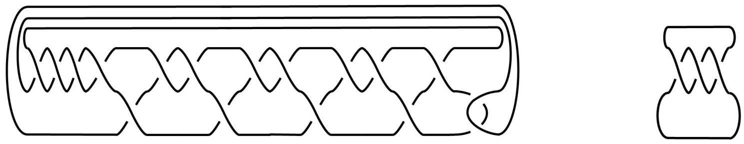


Figure 8: Two equivalent diagrams; the first one is a knot diagram in $\mathcal{D}$ after a crossing change at c; the second one is a diagram of the torus knot $T(2,3)$.

Similarly to the gap between the classical alternation number and the dealternating number, for each positive integer n, there exists a family of infinitely many hyperbolic prime knots with braid alternation number 1 and braid dealternating number greater than or equal to n, whose braid index is $n+3$.

Theorem 4.3. *For each $n \in \mathbb{N}$, there exists an infinite knot family $\mathcal{D}_n$ in $\mathcal{D}$ such that if $K \in \mathcal{D}_n$, then $alt(K) = Balt(K) = 1$ and $Bdalt(K) \geq n$.*

Proof. Let K be a knot in $\mathcal{D}_n$. Lemma 4.2 implies that if K is a knot in $\mathcal{D}_n$, then $alt(K) = 1$. Consequently, $Balt(K) \geq 1$. We will prove that $Balt(K) \leq 1$. Let D be a diagram of K, for instance, the diagram in Figure 7. After a crossing change in 3-tangle c of D, this diagram becomes a knot diagram D', see Figure 8. Hence, D' is a diagram of either a torus knot of two strands or the trivial knot and can be presented as an alternating closed braid. By Theorem 2.1, we know that D can be deformed into a closed braid $\hat{B}$ whose crossings include all crossings of D, an example is shown in Figure 9. Then, $\hat{B}$ after a crossing change in the corresponding crossing of 3-tangle c yields to an alternating closed braid. Therefore, $Balt(K) = 1$. Besides, Lemma 4.2 implies that if K is a knot in $\mathcal{D}_n$, then the dealternating number of K is n. Therefore, due to Inequality (1), we have that $Bdalt(K) \geq n$. □

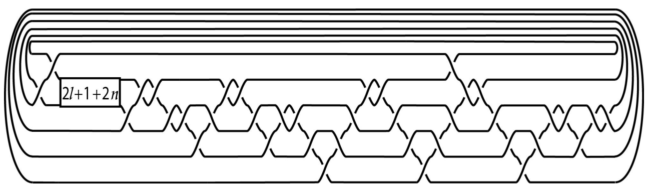


Figure 9: A closed braid diagram of K in $\mathcal{D}$ with $n = 3$.

The second author in [15] defined a distance to the set of fibered links as follows.

Definition 4.4. *The fibering number $f(D)$ of a link diagram D is the minimum number of crossing changes necessary to transform D into a diagram of a fibered link.*

The fibering number of a link L, denoted by $f(L)$, is the minimum fibering number of any diagram of L.

Prime knots up to 10 crossings have fibering numbers less or equal to 1 [10]. However, there are many knots with larger fibering numbers [15].

It follows from their definitions, Theorem 2.1, and Lemma 3.2 that $f(L) \leq Balt(L)$, for any link L. Furthermore, due to Theorem 2.1 and the fact that the trivial link can be presented as an alternating closed braid, the unknotting number of a link L is an upper bound of the braid alternation number of L. Then, for a link L, we have the following relations between these numerical invariants.

$$\begin{array}{ccccc} dalt(L) & \leq & Bdalt(L) & & \\ \vee\!| & & \vee\!| & & \\ alt(L) & \leq & Balt(L) & \leq & u(L) \\ & & \vee\!| & & \\ & & f(L) & & \end{array} \tag{2}$$

Proposition 4.5. *For all $n \in \mathbb{N}$, there exists a knot K such that $Balt(K) = n$ and $alt(K) = 0$.*

Proof. Let K be the n-fold connected sum of a twisted double of the trivial knot with Alexander polynomial $\Delta(L;t) = mt^2 + (1-2m)t + m$ for $m \in \mathbb{N}$ with $|m| \geq 2$. The knot K has both the fibering number and unknotting number equal to n, [15]. Hence, it follows from inequality (2) that $Balt(K) = n$. On the other hand, since the twisted double of the trivial knot is alternating, and the connected sum of alternating knots is alternating, we have that $alt(K) = 0$. □

In particular, the knot 5_2 is a twisted double of the trivial knot with Alexander polynomial $\Delta(L;t) = 2t^2 - 3t + 2$. Then, the n-fold connected sum of the knot 5_2 has alternation number zero and braid alternation number n. Thus, an arbitrarily large gap between the alternation number and the braid alternation number is shown. Since the knot considered is alternating, that gap implies another large gap between the dealternating number and the braid dealternating number. In a similar way, we can obtain a gap between the fibering number and the braid alternation number.

Proposition 4.6. *For all $n \in \mathbb{N}$, there exists a knot K such that $Balt(K) = n$ and $f(K) = 0$.*

Proof. Let K be the n-fold connected sum of the knot 9_{42}. The knot K has alternation number n, [1]. Since the unknotting number of the knot 9_{42} is 1, we can conclude from inequality (2) that $Balt(K) = n$. On the other hand, since the connected sum of fibered knots is fibered [7] and $f(9_{42}) = 0$, we have that $f(K) = 0$. □

Besides, we can obtain a gap between the unknotting number and the braid alternation number.

Proposition 4.7. *For all $l \in \mathbb{N}$, there exist infinitely many knots K such that $Balt(K) = 1$ and $l \leq u(K) \leq l+1$.*

Proof. Let K be a knot of the family $\mathcal{D}$ given at [12], i.e. the closure of the product of the 3-braid $\sigma_2^{2l+1}\Delta^{2n}$ with $l,n \in \mathbb{N}$ and the 3-tangle c. It follows from Lemma 4.2 that the knot K has braid alternation number 1. Let D_+ be the diagram of K with the orientation such that the crossings in the tangle c are positives. After one crossing change at c, we obtain a diagram D_-, which is a diagram of the knot $T(2,2l+1)$. Since $u(T(2,2l+1)) = l$ and $\bar{\sigma}(T(2,2l+1)) = 2l$ for $l \in \mathbb{N}$, it follows from Lemmas 2.2 and 2.3 that $l \leq u(K) \leq l+1$. □

Furthermore, the Alexander polynomial of a knot $K \in \mathcal{D}$ has been obtained in [13]. So, for all $l, n \in \mathbb{N}$, we have that (3) holds.

$$\Delta(K;t) = \left[\sum_{i=1}^{l}(t^{-i}+t^{i})(-1)^{l-i}\right]+(-1)^{l}+(-t^{-l}-t^{l}+t^{-(l+1)}+t^{l+1}) + (t^{-(l+n)}+t^{l+n}-t^{-(l+n+1)}-t^{l+n+1}). \quad (3)$$

In consequence we have that $\Delta(K;-1) = (-1)^{l}(2l+1+4(-1^{n}-1))$. From Lemma 2.2, it follows that $\bar{\sigma}(K) = 2l$ except when $l \leq 3$ and n is odd. Hence, $u(K) = l+1$ if $l \leq 3$ and n is odd.

5 Table of prime knots

In [17], prime links with braid index up to 10 are ordered by employing reduced closed braid diagrams, which have the minimal number of crossings. By using this enumeration, braid presentations in [19], and Lemma 3.3, we discern prime knots K up to nine crossings such that $Balt(K) = 0$, see Table 1.

Note that due to Lemma 3.2, we can determine the knots up to 9 crossings with $Balt(K) \neq 0$. But, Lemma 3.2 is not useful to determine the value of $Balt(K)$ for all knots with 10 crossings or more. In those cases, Lemma 3.3 can be used instead. However, for links with 11 crossings or more, an enumeration of reduced closed braids is needed.

For each prime knot K up to nine crossings with $Balt(K) \neq 0$, we estimate an upper bound of its braid alternation number. In particular, we calculate the $Bdalt(K)$ for all these knots. In the cases of the knots 9_2 and 9_{39}, the unknotting number is used instead of only $Bdalt(K)$. For the knots 9_{35}, 9_{38}, 9_{41}, 9_{48}, and 9_{49}, the braid alternation number is calculated directly from their diagrams.

The invariant $Balt(K)$ has been obtained for all prime knots up to 9 crossings except for $8_3, 9_5$, and 9_{35}. In the remaining cases, other criteria are needed to determine whether these knots are related by a crossing change to knots with $Balt(K) = 0$ and whether $Balt(K) = Bdalt(K)$ for these knots.

Example 5.1. *Previously, we obtained that* $1 \leq Balt(5_2)$. *An upper bound of* $Balt(5_2)$ *can be estimated by using the unknotting number or the braid dealternating number* $Bdalt(5_2)$. *In Figure 10, braids whose closures are the knot* 5_2 *are given; the braid in (c) yields to be alternating after a crossing change. Then* $Balt(5_2) = Bdalt(5_2) = 1$.

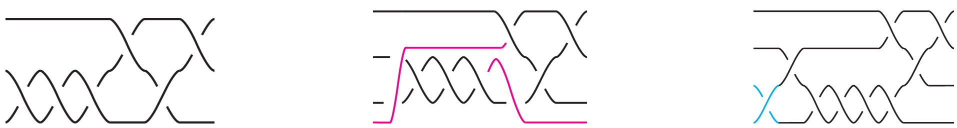


Figure 10: Braids whose closures are the knot 5_2. The last one after a crossing change becomes alternating.

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Knot	alt	Balt	Bdalt	Knot	alt	Balt	Bdalt	Knot	alt	Balt	Bdalt
3_1	0	0	0	8_{15}	0	1	1	9_{22}	0	0	0
4_1	0	0	0	8_{16}	0	0	0	9_{23}	0	1	1
5_1	0	0	0	8_{17}	0	0	0	9_{24}	0	0	0
5_2	0	1	1	8_{18}	0	0	0	9_{25}	0	1	1
6_1	0	1	1	8_{19}	1	1	1	9_{26}	0	0	0
6_2	0	0	0	8_{20}	1	1	1	9_{27}	0	0	0
6_3	0	0	0	8_{21}	1	1	1	9_{28}	0	0	0
7_1	0	0	0	9_1	0	0	0	9_{29}	0	0	0
7_2	0	1	1	9_2	0	1	[1,2]	9_{30}	0	0	0
7_3	0	1	1	9_3	0	1	1	9_{31}	0	0	0
7_4	0	1	1	9_4	0	1	1	9_{32}	0	0	0
7_5	0	1	1	9_5	0	[1,2]	[1,2]	9_{33}	0	0	0
7_6	0	0	0	9_6	0	1	1	9_{34}	0	0	0
7_7	0	0	0	9_7	0	1	1	9_{35}	0	[1,2]	[1,2,3]
8_1	0	1	1	9_8	0	1	1	9_{36}	0	0	0
8_2	0	0	0	9_9	0	1	1	9_{37}	0	1	1
8_3	0	[1,2]	[1,2]	9_{10}	0	1	1	9_{38}	0	1	[1,2]
8_4	0	1	1	9_{11}	0	0	0	9_{39}	0	1	[1,2,3]
8_5	0	0	0	9_{12}	0	1	1	9_{40}	0	0	0
8_6	0	1	1	9_{13}	0	1	1	9_{41}	0	1	[1,2,3]
8_7	0	0	0	9_{14}	0	1	1	9_{42}	1	1	1
8_8	0	1	1	9_{15}	0	1	1	9_{43}	1	1	1
8_9	0	0	0	9_{16}	0	1	1	9_{44}	1	1	1
8_{10}	0	0	0	9_{17}	0	0	0	9_{45}	1	1	1
8_{11}	0	1	1	9_{18}	0	1	1	9_{46}	1	1	1
8_{12}	0	0	0	9_{19}	0	1	1	9_{47}	1	1	1
8_{13}	0	1	1	9_{20}	0	0	0	9_{48}	1	1	[1,2]
8_{14}	0	1	1	9_{21}	0	1	1	9_{49}	1	1	[1,2,3]

Table 1: Prime knots up to 9 crossings with their alternation, braid alternation, and braid dealternating numbers.

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